Quantum Walk on a Spin Network and the Golden Ratio as the Fundamental Constant of Nature

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Abstract

We apply a discrete quantum walk from a quantum particle on a discrete quantum spacetime from loop quantum gravity and show that the related entanglement entropy drives an entropic force. We apply these concepts in a model where walker positions are topologically encoded on a spin network.

Then, we discuss the role of the golden ratio in fundamental physics by addressing charge and length quantization and by analyzing the ratios of fundamental constants—the limits of nature. The limit of minimal length and volume arising in quantum gravity theory indicates an underlying principle that we develop herein.

1 Introduction

One of the principal results from Loop Quantum Gravity (LQG) is a discrete spacetime—a network of loops implemented by spin networks [1] acting as the digital/computational substrate of reality. In order to better understand this substrate, it is natural to use tools from quantum information / quantum computation. Gravity, from a general perspective, has been studied with thermodynamic methods. In recent years, numerous questions on black hole entropy and entanglement entropy have made this an active field of research. In terms of quantum information and quantum computation, advances have been achieved with the aid of many new mathematical tools. Herein, we present the development of one such tool, which we call the discrete-time quantum walk (DQW). We will see that the problem of a quantum particle on a fixed spin network background from LQG can be worked out with the DQW. This gives rise to a new understanding of entanglement entropy and

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entropic force, permitting the proposal of a model for dynamics. In terms of physical ontology, we suggest dynamics and mass emerge from this spin network topology, as implemented by the DQW.

In summary, the first part of this paper is a reinterpretation of results from LQG that emphasizes the quantum information perspective of a quantumgeometric spacetime. That is, it adopts Wheeler's *it from bit* and the newer *it from qubit* ontologies—the general digital physics viewpoint.

One important view of reality is the digital physics paradigm—the idea that reality is numerical at its core [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In the second part of this paper, we defend the conjecture that reality is information-theoretic at its core by presenting experimental and mathematical justification of our conjecture that the golden ratio is the fundamental dimensionless constant of nature. Numbers include shape-symbols, such as simplex-integers [14]. Generally, a symbol is an object that represents itself or another object. Symbols can be self-referential and participate in selfreferential codes or languages, such as quasicrystals [14, 2, 15]. Generally, symbols are highly subjective, where meaning is at the whim of language users. However, simplexes¹, as geometric number and set theoretic symbols which represent themselves, have virtually no subjectivity. That is, their numeric, set theoretic and geometric meaning is implied via geometric first principles. If space, time and particles were patterns in a quantized geometric code, low subjectivity geometric symbols, such as simplex-integers, could serve as both spatiotemporal and numerical quanta of space while also being part of an organized code or language called a quasicrystalline phason code. One element of digital physics we discuss here is the discreteness of spacetime and charge in general—something that a geometric code is well suited to provide.

The *voxel* of space is an old idea that, today, has acquired a body of experimental evidence as well as mathematical proofs [16]. Its early construction starts with the birth of quantum field theory, where it was noticed that models become more regular with a cutoff. When one considers quantum gravity, like Bronstein did in 1936 [17], this question becomes fundamental because, as opposed to smooth spacetime field theories, gravity does not allow an arbitrarily high concentration of charge in a small region of spacetime. Bronstein concluded that this leads to an inevitable limit of the precision to which one can measure the strength of the gravitational field. Advances on different fronts in the following years lead to a general understanding that this *voxel* of space, a minimal volume, must be the foundational building block of any realistic proposal of quantization of the gravitational field. Such deductions provide support for the digital physics view. Quantum mechanics indicates that nature must have a minimum length, volume and time. We can cite the rigorous developments of string theory, LQG, the Maldacena conjecture, black hole physics, various thought experiments, the generalized uncertainty principle and many other works as support of the discretized

¹An n-simplex is an n-dimensional polytope which is the convex hull of its n+1 vertices. For example the triangle in 2D, tetrahedron in 3D, 4-simplex in 4D...

spacetime view. For a review, references and historical details, we recommend reference [16]. The discrete nature of quantum gravity theories implies that time and motion are ordered numerical sets on point-like substructures, which can be viewed as discrete pre-spacetime graphs acting as possibility spaces for graph actions. Upon such graph drawings (a geometric representation of a graph), a quantum gravity code theoretic formalism can express itself non-deterministically. This approach leads to a new ontology for the substrate of reality that may replace older ontologies based on either randomness or determinism. The new ontology is called the code theoretic axiom [2], where a finite set of irreducible geometric symbols and a finite set of nondeterministic syntactical rules express as code legal and physically realistic manifestations. Self-organizing expressions of a quantum code are inherently non-deterministic because codes/languages are non-deterministic.

Herein, we address implications of minimal volume, the quantum of charge of the gravitational field (along with minimal charge in general) in light of the code theoretic view. The experimentally validated result of electric charge quantization has a standard theoretical explanation in Dirac magnetic monopoles. This field, which applies topological arguments to address charge quantization, follows Dirac's work in general [18]. We go further here to understand that there are underlying principles governing charge quantization. This is revealed when one considers the Planck volume limit of space. One consequence we discuss is the special role of the golden ratio, where we show that fundamental limits of nature can be understood in terms of the golden ratio.

2 Methods

2.1 Particle interacting LQG

We start by considering a quantum particle on a quantum gravitational field from LQG [19]. In LQG, spin networks define quantum states of the gravitational field. To consider a quantum particle on this gravitational field, we consider the state space built from tensor product of the gravity state \mathcal{H}_{LQG} and the particle state \mathcal{H}_P , $\mathcal{H} = \mathcal{H}_{LQG} \otimes \mathcal{H}_P$. The LQG state space can be spanned by a spin network basis $|s\rangle$ that is a spin network graph $\Gamma = (V(\Gamma), L(\Gamma))$, with $V(\Gamma)$ vertices coloring which elements denoted by $v_1, v_2...,$ and $L(\Gamma) \rightarrow l_1, l_2..., (\{\frac{1}{2}, 1, \frac{3}{2}...\})$ links coloring. For the particle state space the relevant contribution comes from the position on the vertices of graph Γ , spanned by $|x_n\rangle$ where (n = 1, 2...). The quantum state of the particle captures information from discrete geometry and cannot be considered independently from it. Therefore the Hilbert space is spanned by $|s, x_n\rangle$ and the Hamiltonian operator can be derived by fixing a spin network to the graph, and calculating the matrix element of this operator $\langle \psi | H | \psi \rangle$ with $\psi \in H_P$. Accordingly, for the interaction between the particle and the fixed gravity state space [19] we have,

$$\langle \psi | H | \psi \rangle = \kappa \sum_{l} j_l (j_l + 1) \left(\psi(l_f) - \psi(l_i) \right)^2, \tag{1}$$

where κ is a constant that we can initially take as equal to one, l_f are the final points of the link l and l_i the initial points. The interaction term takes this form because the relevant Hilbert space depends on the wave functions at the vertices. So if the link l starts at vertex m and ends at vertex n we can change the notation, relabeling the color of this link l between m and n as j_{mn} and the wave function on the end points as $\psi(v_m)$, $\psi(v_n)$. Now, for H, we have

$$\langle \psi | H | \psi \rangle = \kappa \sum_{l} j_{mn} (j_{mn} + 1) \left(\psi(v_n) - \psi(v_m) \right)^2, \qquad (2)$$

The operator H is positive semi-definite. The ground state of H corresponds to the case where the particle is maximally delocalized. This leads to entropy [19]. In reference [20] it was considered that the classical random walk is associated with (2), a Markov chain. The transition probabilities for this random walk are

$$P_{mn} = \frac{j_{mn}(j_{mn}+1)}{\sum_{k} j_{mk}(j_{mk}+1)}.$$
(3)

and [20] shows that this reproduces an entropic force. This random walk is implemented with the Laplacian solution (2) and differs from the Laplacians coming from the discrete calculus work out in [21]. We will consider DQW relations with more general Laplacians in future work.

2.2 DQW

As study the influence of discrete geometry on a quantum field. Here, it is natural to consider the quantum version of random walks. To have DQWs we need an auxiliary subspace, the coin toss space, to define the unitary evolution

$$U = S(C \otimes I), \tag{4}$$

where S is a swap operation that changes the position to a neighbor node and C is the coin toss operator related to this auxiliary subspace. On some graphs its clear how to define the coin toss Hilbert space [22] but for the spin network considered above its not clear exactly what auxiliary space to use. For a more general approach, we can make use of Szegedy's DQW [23, 24], which we can utilize in two ways: 1- consider a bipartite walk or 2 - a walk with memory. For a bipartite walk, if we consider a graph Γ , we can simply make an operation of duplication to obtain a second graph $\bar{\Gamma}$. For our purpose here, it is better to consider the second option – the walk with memory, considering the N_v -dimensional Hilbert space \mathcal{H}_n , $\{|n\rangle, n = 1, 2, ..., N_v\}$ and \mathcal{H}_m , $\{|m\rangle, m = 1, 2, ..., N_v\}$, where N_v is the number of the vertex $V(\Gamma)$. The state of the walk is given as the product $\mathcal{H}_n^{N_v} \otimes \mathcal{H}_m^{N_v}$ spanned by these bases. That is, by states at the previous $|m\rangle$ and current $|n\rangle$ steps, defined by

$$|\psi_n(t)\rangle = \sum_{m}^{N_v} \sqrt{P_{mn}} |n\rangle \otimes |m\rangle, \qquad (5)$$

where P_{mn} is the transition probabilities that define a classical random walk, a Markov chain, which is a discrete time stochastic process without a memory, with

$$\sum_{n} P_{mn} = 1. \tag{6}$$

Note that (6) is implied by definition of (3). For the evolution, we can consider a simplified version of Szegedy's DQW [25] by defining a reflection, which we can interpret with the unitary coin toss operator²,

$$C = 2\sum_{n} |\psi_{n}\rangle \langle\psi_{n}| - I, \qquad (7)$$

and a reflection with inverse action of the P swap (previous and current step) that can be implemented by a generalized swap operation

$$S = \sum_{n,m} |m,n\rangle \langle n,m|, \qquad (8)$$

where we have the unitary evolution

$$U = CS, (9)$$

that defines the DQW.

So using Szegedy's approach is a straightforward way to obtain the discrete quantum walk on the spin network Γ considered above. It is given by equations (5 - 9) with P given by (3). Namely, for equation (5)

$$|\psi_n(t)\rangle = \sum_{m=0}^{N_v} \sqrt{\frac{j_{mn}(j_{mn}+1)}{\sum_k j_{mk}(j_{mk}+1)}} |n\rangle \otimes |m\rangle.$$
(10)

We can interpret the coin toss space as the space of decisions encapsulating a nondeterministic process possessing memory, such that we have a unitary evolution. This approach makes the usual entropy convert into entanglement entropy between steps in time. Szegedy's DQW is a general algorithm quantizing a Markov chain defined by transition probabilities $P_{n,m}$. These transition probabilities are obtained directly from the Hamiltonian of the system such that Szegedy's DQW is used to simulate this system with quantum computation [26].

 $^{^2\}mathrm{Szegedy's}$ DQW does not generally use a coin toss operator in the literature

2.3 Entanglement Entropy and Entropic Force

We turn now to calculate entanglement entropy. Consider the Schmidt decomposition. Take a Hilbert space \mathcal{H} and decompose it into two subspaces \mathcal{H}_1 of dimension N_1 and \mathcal{H}_2 of dimension $N_2 \geq N_1$, so

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2. \tag{11}$$

Let $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$, and $\{|\psi_i^1\rangle\} \subset \mathcal{H}_1, \{|\psi_i^2\rangle\} \subset \mathcal{H}_2$, and positive real numbers $\{\lambda_i\}$, then the Schmidt decomposition would be

$$\left|\psi\right\rangle = \sum_{i}^{N_{1}} \sqrt{\lambda_{i}} \left|\psi_{i}^{1}\right\rangle \otimes \left|\psi_{i}^{2}\right\rangle, \qquad (12)$$

where $\sqrt{\lambda_i}$ are the Schmidt coefficients and the number of the terms in the sum is the Schmidt rank, which we label N. With this, we can calculate the entanglement entropy between the two subspaces

$$S_E(1) = S_E(2) = -\sum_{i \in N} \lambda_i log\lambda_i.$$
(13)

We can now calculate the local entanglement entropy between the previous m step and the current n step (similar for current and next steps). Identifying the Schmidt coefficients $\sqrt{\lambda_i}$ with $\sqrt{P_{mn}}$, with P_{mn} given by (3) and, from (10), we see that the Schmidt rank N is the valence rank of the node. Then insert in (12, 13), and the local entanglement entropy on current step is

$$S_{E_n} = -\sum_m^N P_{mn} log P_{mn}.$$
 (14)

By maximizing Entanglement Entropy

$$S_{E_n} = \log N_{max},\tag{15}$$

where N_{max} is the largest valence. Which gives the entropic force for gravity worked out in [20, 27] with $\frac{dS}{dx} = |S_{E_n} - S_{E_m}|$ proportional to a small number identified with the particle mass M

$$\frac{dS}{dx} = |S_{E_n} - S_{E_m}| = \alpha M, \tag{16}$$

where, if we take the logarithm of (15) to be base 2, α is a constant of dimension [bit/mass].

Each of these interpretations has related applications. For example, with the DQW, we have unitary evolution by encoding a non-deterministic part of the classical Markov chain. This gives an internal structure for the particle as well as the entanglement entropy value. In such a digital physics substrate, the particle walks in such a way as to maximize its entanglement entropy based upon the inherent memory of its walking path (a measurement of the system on n implies the state at previous m), which generates an entropic force.

Interestingly, we can consider von Neumann Entropy (14) in the context of quantum information. The probabilities are the Markov chain connecting the steps. Accordingly, the particles construct "letters" of a spatiotemporal code as expression of the allowed restricted but non-determined walking paths, where the entropy measures the information needed. From this, we see how the entropic force emerges.

3 Results

3.1 Entropy of Black Hole

The framework above is a discretized quantum field expressing phases as algebraically allowed patterns upon fixed discrete geometry. Specifically a moduli space algebraic stack. Gravitational entropic force suggests a unified picture of gravity and matter via a quantum gravity approach. Consider now a regime from pure quantum gravity, like the black hole quantum horizon, so that there are no quantum fields but only quantum geometry. We can use our DQW to simulate this regime.

Again, DWQs encode local entanglement entropy (14) in the sense that the chains or network of step paths are always dynamically under construction as a "voxelated" animation. This resembles the isolated quantum horizons formulation of LQG [28, 29, 30, 31, 32], which gives explains the origin of black hole entropy in LQG. In this scenario, the Horizon area emerges from the stepwise animated actions at the Planck scale that are simulated by DQWs. We argue that DQWs are the Planck scale substrate forming the emergent black hole quantum horizon, where particle masses composite to black hole mass in the full aggregate of quantum walks on the spin network of the event horizon.

In the isolated quantum horizons formulation, entropy is generally calculated by considering the eigenvalues of the area operator A(j) and introducing an area interval $\delta a = [A(j) - \delta, A(j) + \delta]$ of the order of the Planck length with a relation to the classical area a of the horizon. A(j) is given by

$$A(j) = 8\pi\gamma l_p^2 \sum_{l} \sqrt{j_l(j_l+1)},$$
(17)

where γ is the Barbero-Immirzi parameter and l_p the Planck length. The entropy, in a dimensional form, is

$$S_{BH} = lnN(A), \tag{18}$$

with N(A) the number of micro-states of quantum geometry on the horizon (area interval a) implemented, combinatorially, by considering states with link sequences that implement the two conditions

$$8\pi\gamma l_p^2 \sum_{l=1}^{N_a} \sqrt{j_l(j_l+1)} \le a,$$
(19)

related to the area, where N_a is the number of admissible j that puncture the horizon area and

$$\sum_{l=1}^{N_a} m_l = 0 \tag{20}$$

related to the flux with m_l , the magnetic quantum number satisfying the condition $-j_l \leq m_l \leq j_l$.

The detailed calculation [29, 32] shows that the dominant contribution to entropy comes from states in which there is a very large number of punctures. Thus, it is productive to interpret this entropy as quantum informational entropy (14). Let us investigate how the horizon area and related entropy emerges from maximal entanglement entropy of DQWs. Condition (19) is associated with these DQWs. From (14 and 15), considering edge coloring, maximal entanglement entropy occurs for states on nodes of large valence N_{max} and sequence with $j_l = \frac{l}{2}$ ($l = 1, 2, ..., N_{max}$). Accordingly, for the entropy calculation, it is admissible that j, which punctures the horizon area, respects these sequences and that associated nodes have large valence rank. Therefore, we can rewrite condition (19) as

$$\sum_{i=1}^{N_{a_i}} a_i = a_c, \tag{21}$$

where N_{a_i} is the number of admissible nodes with

$$a_c = \frac{a}{4\pi\gamma l_p^2}.$$
(22)

and each a_i is calculated from

$$a_i = \sum_{l=1}^{N_{max}} \sqrt{l(l+2)},$$
(23)

and considering the dominant contributions given by the over-estimate each a_i and counting $N(a_c)$ that will give N(A), each a_i is an integer strictly greater than 1

$$\sqrt{l(l+2)} = \sqrt{(l+1)^2 - 1} \approx l+1,$$
 (24)

which means that the combinatorial problem we need to solve is to find $N(a_c)$ such that (21) holds. This was discussed in a similar problem in [28]. It is straightforward³ to see that

$$logN(A) = \frac{log(\phi)}{\pi\gamma} \frac{a}{4l_p^2},$$
(25)

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio⁴.

³Because the cardinal $N(a_c)$ of the set of ordered tuples of integers strictly greater than 1 summing to a_c is the a_c^{th} Fibonacci number $F(a_c)$. ⁴The golden ratio [33] is an irrational number (1.61803398875...) as the solution to the

⁴The golden ratio [33] is an irrational number (1.61803398875...) as the solution to the equation $\phi^2 = \phi + 1$.

With this established, we can now conjecture that this entropy, relating to states that give maximal entanglement entropy, is simply the entanglement entropy of the DQWs. Accordingly, holographic quantum horizons can be simulated by the DQWs with maximal internal entanglement entropy. The walker moves are from node m to node n, each each with large valence number so that equation (16) works. If we consider a walker mass spanning I with Planck mass m_p , we can choose the proportionality constant so that

$$\Delta S_E = |S_{E_n} - S_{E_m}| = I, \tag{26}$$

so $M = Im_p$ gives the amount of information on the horizon, and $\triangle S_E$ is given in *bits*. We can make explicit the log_2 on (25) so that it is given in *bits* too and propose that $log_2N(A) = \triangle S_E$ and that this measure of information gives the emergent Bekenstein-Hawking entropy. DQWs encode or express black hole information, so to infer its entropy, one needs to make contact with some frame of DQWs. The particles that simulate black holes will be more probable on a node of maximal entanglement entropy encapsulating logN(A). Note that logN(A) is not the usual statistical entropy because we have not taken into account all of the micro-states or the flux condition. From (18) the black hole entropy, changing the logarithm, is

$$S_{BH} = log_2 N(A) = \frac{log_2(\phi)}{\pi\gamma} \frac{a}{4l_p^2}.$$
 (27)

Therefore, Bekenstein-Hawking entropy is recovered by setting the Barbero-Immirzi parameter

$$\gamma = \frac{\log_2(\phi)}{\pi},\tag{28}$$

showing that the states of maximal entanglement entropy are dominant in black hole entropy. So we can think of Bekenstein-Hawking entropy as emergent from the local entanglement entropy above. A horizon area a spanning 4I Planck areas has I bits like (26).

3.2 A model of walker position topologically encoded on a spin network

The Clebsh-Gordan condition at each node is realized by covering the graph with loops. From (3) and (14), we can compute the local entropy from a vertex as

$$S_{E_n} = \log\sigma - \frac{1}{\sigma} \sum_{m}^{N} j_{mn}(j_{mn} + 1) \log\left(j_{mn}(j_{mn} + 1)\right), \qquad (29)$$

where $\sigma = \sum_{m}^{N} j_{mn}(j_{mn}+1)$ of neighbor links. For example, at a node $\{2,3,3\}$, $j = \{1, \frac{3}{2}, \frac{3}{2}\}$ gives $\sigma = \frac{19}{2}$ and $S_{E_n} = 1.06187$. At a node $\{2, 2, 2\}$, $j = \{1, 1, 1\}$ gives $\sigma = 6$, $S_{E_n} = 1.09861$, which is the maximum possible local



Figure 1: Loops.

entropy. Note that (29) is the local entropy formula given in [20] as 14 and is in accordance with well known LQG formulas for quantized length and area. See figure (1).

The local entropy at each node is color coded. From equation (16), a massless particle moves on the same color and a massive particle moves along constant absolute color differences.

In the figure (2) : (blue) $\{l_1, l_2, l_3\} = \{0, 1, 1\}$ or $\{0, 2, 2\}$ (side effect) (white) ... $\{1, 2, 3\}$ (yellow) ... $\{1, 1, 2\}$ (orange) ... $\{2, 3, 3\}$ (red) ... $\{2, 2, 2\}$.

Dynamics:

Photons can orbit on orange hexagons. No possible particles with constant mass. Possible travel of massive particle, with $mass = |S_{(red)} - S_{(orange)}|$, (= 0.037) interacting with photons at some orange vertex.



Figure 2: Entropy is color coded.

The walker position or the presence of a particle at one node is encoded by a triangle. Its move is a couple of 3-1 and 1-3 Pachner moves on neighbor positions, piloted by the walk probability. See figure (3).



Figure 3: Particle and Pachner moves.

4 Ontological discussion

4.1 Minimal volume and charge

The generic implication of quantum gravity, the existence of a minimal volume (v_o) near the Planck scale indicates a new fundamental limit in nature. This fundamental limit that arises during the quantization of gravitational fields is similar to the fundamental charge q_e that arises in the quantization of electromagnetic fields⁵. For all these fundamental limits, there is one fundamental dimensionless constant, which relates them. It is often said that the dimensional constants are more fundamental than the dimensionless constants [34]. This is especially true when they are fundamental building blocks of a theory, such as occurs when there are general limits found in nature such the speed of light c in special relativity and the minimal action \hbar in quantum mechanics. The dimensionless constants depend on the model, such as the standard model of particle physics, and they are generally determined experimentally, where there is a meaningful experimental margin of error. We argue that there are dimensionless constants of nature that have a special and fundamental role as ratios of the fundamental limits of nature.

In quantum electrodynamics (QED), the fundamental ratio is the finestructure constant α , wherein the relation is obtained from Dirac's quantization of magnetic charge [35]

$$\alpha = \frac{e}{g}n,\tag{30}$$

and where n is some integer, g is the magnetic charge rewritten in Coulombs and e the elemental electric charge. The magnetic charge is related to the

⁵The same discussion can be made for fundamental charges associated with electroweak and the strong force that is in agreement with the understanding of a fundamental running coupling constant using renormalization group equations.

non-trivial topology of charge space. Or, considering a discrete substrate, for each lattice we use to describe the electric field, there is a dual lattice corresponding to magnetic flux. So, for each site, we have the two fundamental charges living in respective dual spaces to one another. The charges of QED are quantized such that the ratio between any two is α , and the coupling constant is related to the strength of the force. We will not elaborate further here. However, we note that the monopole magnet provides a new understanding of the relation between topology, geometry and physics that is captured in the simplest form by equation (30).

For more insight, we can use a similar formula without flux by fixing to the Planck scale,

$$\alpha = e^2 \frac{k_e}{\hbar c} = \left(\frac{e}{q_p}\right)^2 \tag{31}$$

where k_e is Coulomb's constant or the electric force constant, a constant from the QED interaction, \hbar is the reduced Planck's constant, c is the speed of light in vacuum and q_p is the Planck charge. In terms of α^{-1} ,

$$\alpha^{-1} = \frac{1}{e^2} \frac{\hbar c}{k_e} = \left(\frac{q_p}{e}\right)^2. \tag{32}$$

The approximate value for this constant is based on experiments and QED perturbation theory calculations. The CODATA [36] recommended value is

$$\alpha = 0.0072973525664 \tag{33}$$

and

$$\alpha^{-1} = 137.035999139 \tag{34}$$

and is associated with the QED scale. It it important to take note of the fact that the CODATA value of α^{-1} in equation (34) is based on an average of disagreeing results given by five different experimental techniques for the measurement of Planck's constant. To explain, α is a ratio relative to the elementary charge. CODATA does not independently define the quantity for α . Instead, a value is derived from the relation (31). Accordingly, the accuracy of measurement of Planck's constant is proportional to our knowledge of the fine structure constant. The CODATA value for Planck's constant is a weighted average of the following five experimental techniques that agree only at the 4th place after the decimal ($6.6260 \times 10^{-34} Js$). See figure 4 [36].

Experimentally, we can therefore only be confident in the value of h to the 4th decimal place, which limits our knowledge of what α is to the 4th decimal place. If one wishes to assume a probable value greater than the 4th decimal place, it would be logical to use the most advanced and precise experimental technique to date, as opposed to the CODATA approach that uses older lower resolution results to down-grade the precision of the more precise newer results. In the case of the gravitational constant, the precision is only to the 2nd decimal place. The most advanced high energy technique is based on atomic scale experiments not available when the earlier techniques

| Method | Value of <i>h</i> (10 ^{−34} J⋅s) |
|----------------------------------|--|
| Watt balance | 6.626 068 89(23) |
| X-ray crystal density | 6.626 0745(19) |
| Josephson constant | 6.626 0678(27) |
| Magnetic resonance | 6.626 0724(57) |
| Faraday constant | 6.626 0657(88) |
| CODATA 2010 recommended value | 6.626 069 57(29) |

Figure 4: Planck's constant. CODATA 2010.

averaged into the current CODATA value were developed. Specifically, the most technologically advanced calculation for measuring the gravitational constant was reported in Nature in 2014 [37]. It is a result derived by probing the atomic scale. Herein, we will refer to this as "highest resolution value of G", denoted as G_{HR} , $G_{HR} = 6.671 \times 10^{-11} m^3 kg^{-1} s^{-2}$.

In an attempt to discover what the actual minimum volume v_o is that nature uses, there have been many attempts to quantify it from string theory [38], loop quantum gravity, (LQG) [1] and others [16]. However, none of these unification theories have made successful predictions. So until a predictive quantum gravity theory is discovered, one cannot make solid theoretical assumptions using the CODATA value beyond a few decimal places for any fundamental constant such as α . When precision is desired, one would want to use G_{HR} combined with c to generate \hbar and, from there, to generate the rest of the constants [39].

There is consensus that the quantum gravity scale is at or near the Planck volume. However, the limit of our knowledge of the value of \hbar directly limits our knowledge to the current limit, which is G_{HR} . Fortunately, actual results permit us to understand that the gravitational constant, G, ties the three following fundamental limits of nature together to build a dimensionless fundamental constant that we call β , which relates to α . The three dimensional constants associated with these limits are the maximal local physical velocity, the speed of light c, the minimal action (or a minimal amount of information) \hbar and this newly discovered minimal volume v_o . It is the fundamental dimensionless ratio defining the relationship between these fundamental dimensional limits. We write this as

$$\beta = \frac{1}{v_o^2} \left(\frac{\hbar G}{c^3}\right)^3 = \left(\frac{v_p}{v_o}\right)^2 \tag{35}$$

or in the inverse form

$$\beta^{-1} = v_o^2 \left(\frac{c^3}{\hbar G}\right) = \left(\frac{v_o}{v_p}\right)^2,\tag{36}$$

where we use the Planck volume, $v_p = l_p^3$, with l_p the Planck length that defines the Planck scale and Planck units as

$$l_p = \sqrt{\frac{\hbar G}{c^3}}.$$
(37)

We will discuss in the next sections how these constants β and α are deeply related to the golden ratio.

For more insight, let us focus on the quantization of the gravitational field using results from LQG [1]. Quantization is performed using a generalization of the lattice of QED – a triangulation of spacetime and its dual two-complex. Here we a split of the gravitational "charge" into two components; one related to flux⁶ (gravitational analog of magnetic field) with area eigenvalues $l_p^2 \sqrt{j(j+1)}$ where j are non-negative half integers. And the other is the gravitational analog of the electric field in the dual space that is the geometry of quantized spacetime, which we will call l_l

$$l_l^2 \propto \gamma l_p^2 \sqrt{j(j+1)},\tag{38}$$

where γ is the Barbero-Immirzi parameter. The proportionality constant in this reference is 8π . From this quantization condition, we derive the eigenvalue of the volume to be

$$v_0^2 \propto \gamma^3 \left(\frac{\hbar G}{c^3}\right)^3,$$
 (39)

which gives

$$\beta \propto \frac{1}{\gamma^3}.$$
 (40)

The proportionality constant in this reference is $6\sqrt{3}/(8\pi)^3$. The theory has some ambiguities and does not permit us to calculate this constant or γ . There are no experimental solutions possible yet. However, black hole calculations based on the limit of general relativity and quantum mechanics provide an approximate value for γ between 0 and 1. See equation (28) and reference [32].

⁶A less discussed quantity in the literature which is used for dynamics in quantum gravity formalisms are the volume flow rate, well known in hydrodynamics and magneto hydrodynamics, related to constant $Q_{\Phi} = \frac{4\pi h G}{c^2}$ [40] given in the SI units by $m^3 s^{-1}$. Following the discussion above about the most accurate value of Planck's constant based on the most precise experimental technology, we reiterate that the CODATA values agree only at the 4th place after the decimal and the value for G agrees only to the 2nd place after the decimal. Using the result of the most recent and high energy atomic scale experimental techniques, G_{HR} , together with the value of h only to the 4th place after the decimal gives $Q_{\Phi} = 0.6180382 \times 10^{-59} m^3 s^{-1}$. This results in a value going out to the 5th decimal place

This indicates a relationship between β and the quantum gravity coupling constant, γ , associated with the gravitational constant. That is, β leads to the precise quantum gravity coupling constant—also a number between 0 and 1.

Correspondence between the two scales, QED and quantum gravity, is expected in a correct unification theory. Our interpretation is that it is related to the quantization of charge and is manifest in the ratios of equations (30)and (38). The result suggests an underlying principle governing this fundamental ratio. We suggest this principle is related to symmetry and graph network theory efficiency, where nature exists as a maximally efficient graph theoretic topological quantum code due to its requirement of classic efficiency as exemplified in the principle of least action and other conserved symmetries ubiquitous in nature, such as the gauge symmetry unification of particles and forces in the standard model of particle physics. Put differently, presume the digital physics view is true, where nature is information theoretic. That is, nature is based on symbolic language and not merely described by symbolic language. From the probability plot in 3-space of the quantum wave form, to gravity theory to classic physics, this supposed language is simulating or expressing itself geometrically in the form of physics. Accordingly, a geometric code would be a good conjecture for what form of symbols would be used in the mathematical language of our geometric universe. And because nature appears to be concerned with efficiency, a maximally efficient geometric code would be logical. Graph drawing codes, such as phason algebraic codes in quasicrystals, may be optimally efficient in the universe of codes capable of expressing a geometric physical universe.

The efficiency of a code, such as a geometric spatiotemporal code, can be ranked by the ratio of binary actions needed to express a given physical system. A geometric based quantum gravity plus particle physics code would express spatiotemporal information using the minimum quantity of irreducible geometric symbols. Trivially, the minimum number of symbols in any code is two. Fundamental quasicrystals such as the Elser-Sloane quasicrystal [41] with H_4 symmetry, derived by projective transformation of a minimal slice of the E_8 lattice, or the quasicrystalline spin network (QSN) [42, 43] with H_3 symmetry are constructed as geometric languages composited from geometric spatiotemporal symbols as points separated by two distances related as 1 and $1/\phi$. Such codes exist physically in nature as quasicrystalline phases of matter, where generally a network of Fibonacci chains [15] in 3-space manifests as energy wells separated by 1 and $1/\phi$ instead of abstract points in the mathematically ideal abstract analogue. At any given moment, some energy wells are occupied by an atom and some are vacant. Over the time domain, the atoms are known to tunnel at very low energy according to a phason code that is inherently non-local.

The binary efficiency of a network can be ranked by the connectivity rank of its nodes—specifically the average valence value of its vertices. Maximally connected graph drawings can only be achieved via the use of the golden ratio, due to it being the only ratio possessing the fractally self-similar quality



Figure 5: To optimally network 1D two-letter codes in 2, 3 and 4 dimensions, we must use the Fibonacci chain with the two letters as $1/\phi$ or the connectivity breaks down. An infinitesimal deviation from the golden ratio spacing renders the object (1) not a quasicrystal because arbitrary closeness of nodes will exist and (2) a non-code because there will be an infinite number of 1D symbols as lengths and (3) not a quantum topological network because non-local influence where an action on one point influences other non-local point on/off states breaks down. We know that the reason for this is because, in the universe of all ratios closed under multiplication and division, only ϕ possesses fractal self-similarity. The above diagram on the left shows a network of five short Fibonacci chains, where nodes overlap perfectly due to the fractal self-similarity of ϕ . To left, we see how a small deviation from the golden ratio spacing destroys the connectivity rank by lowering the average near neighbor valance magnitude and introducing arbitrary closeness between nodes.

Our conjecture is that this fundamental dimensionless constant β discussed above is $\sqrt{\beta} = 1/\phi^3$, which makes the ratio between the Planck length and the minimum length l_o , related to v_o , ϕ and allows for networks where nearest neighbor lengths are restricted only to 1 and ϕ , starting at the Planck scale. Due to its power of network code efficiency, this fundamental ratio would be the building block of the standard model of particle physics and general relativity in a geometric graph-drawing code theoretic unification formalism. Note that in a gauge unification picture, Lie groups, Lie algebras and their associated lattices, such as the E_8 lattice, play a central role [44, 45]. Grand unified theories are elegant and successful. They embed the Standard Model Lie group inside larger groups, where in 6D the largest group possible is E6 [38, 46]. Going to higher dimensions, E6 is a subspace of $E8^7$. E8 is a powerful tool unifying gravity and the standard model via the most popular and foundational form of string theory, heterotic string theory [38, 47, 48] or E8 Grand Unification by itself [49, 50, 51, 52, 53]. Remarkably, E8 symmetry has been found in condensed matter physical experiments [54]. Furthermore, one can recover E8 gauge symmetry physics from an icosian construction of the E_8 lattice $[55]^8$.

 $^{^{7}}SU(3) \times E6$ the maximal sub algebra of E8.

⁸Note that geometrically, E_6 is a sub-lattice embedded in E_8 .

Summarizing this section, if nature is a geometric code, such as a quasicrystal phason code derived from the higher dimensional E_8 lattice, this might explain charge quantization in nature and shed light on a universally generalized optimization principle. Let us discuss this in detail next.

4.2 The principle behind fundamental ratios

To discover the correct quantum gravity and particle physics unification theory, new insights and fundamental principles must be realized [56, 57]. The code theoretic axiom may lead us in the right direction [2]. The existence of a minimal volume implies violation or generalization of principles such as Lorentz transformations, the uncertainty principle and the equivalence principle [58, 59, 60, 61]. We argue for a new principle associated with this minimal volume, a dimensionless constant.

It is trivially true that ratios are more fundamental than numerical values based on arbitrary metrics. If we consider a model of quantum geometry, as developed in various quantum gravity theories, these ratios will be geometric as will any building blocks of a realistic model. If this is the case, what is the principle governing the organization of these building blocks at the Planck scale? Or more generally, what is the principle behind charge and, equivalently, length and volume quantization that can be deduced by looking at the ratios of fundamental limits:

$$v_p = \sqrt{\beta} v_o \tag{41}$$

and,

$$e = \sqrt{\alpha}q_p \tag{42}$$

In Planck units:

$$v_p = 1$$

$$v_o = 1/\sqrt{\beta}.$$
(43)

For the elementary electric charge

$$q_p = 1$$

$$e = \sqrt{\alpha}.$$
(44)

So this building block ratio for charge and volume quantization is a specific fundamental ratio based on the efficiency and symmetry principle we are looking for.

It is generally agreed that fundamental physics relies on a fundamental optimization principle, like we referenced in [62]: "The whole of theoretical physics (classical and quantum) relies on a fundamental optimization principle from which the basic equations of physics can be constructed under the form of Euler–Lagrange equations, relating fundamental quantities, such as energy, momentum and angular momentum. This principle of least action becomes a geodesic principle in the framework of relativity theories (i.e.,

the action is identified with the proper time). The geodesic principle states that the free trajectories are the geodesics of spacetime. It plays a very important role in a geometric relativity theory, since it means that the fundamental equation of dynamics is completely determined by the geometry of spacetime and therefore does not need to be set down by an independent equation. Moreover, in such a framework, the action can be identified (modulo a constant) with the fundamental metric invariant, which is nothing but the proper time itself. The action principle becomes nothing else than the geodesic principle. As a consequence, its meaning becomes very clear and simple: the physical trajectories are those, which minimize the proper time itself".

So the principle of least action is a principle related to efficiency and optimization of resources and has a defined meaning at a classical level; the minimal path. That is, the action is taken as a stationary value. Here we have a clear first-principles understanding of "efficiency". A similar optimization principle can be found in computational systems in the principle of least computational action [63]. Putting together the least action and relativity principles, we are lead to a principle of maximally efficient ratio. However, we are interested in discrete spacetime implied by quantum gravity. Accordingly, at the quantum level, the classical notion of a sharp trajectory followed by a physical system is rejected, leaving an opportunity. Specifically, the ratios defining the discrete geometry at the level of quantum gravity. To make this principle of maximally efficient ratio precise, we will define the discrete substrate of quantum gravity. Before doing so, let us articulate the principle in the context of discrete quantum geometric evolution:

 \hookrightarrow The evolution of a system is such that it uses the most efficient ratio relating its indivisible building blocks.

In section 2.1, we considered a model for a particle interacting with a discrete substrate of the gravitational field, a fixed spin network from LQG. The interaction is given by equation (1). The interaction term takes this form because the relevant Hilbert space depends on the wave functions at the vertices. From this, we derive the dynamics by implementing a unitary evolution with a quantum random walk. Here we will go further to propose that the transition from a node v_m to a node v_n is guided by one action $S(v_n, v_m)$. Equation (1), in fact, implies a Laplacian operator for the graph because we have a discrete structure that derives energy values. Classically, this action is a function that integrates the two steps leading from v_m to v_n , i.e., between the two discrete times. In order to account for such a discrete substrate, the action must be considered probabilistically. From random walks on a graph theoretic spin network, we derive the probability $P(v_n)$ for a particle ending up on an arbitrary vertex v_n along with the transition probabilities $P(v_n, v_m)$ for transitions from v_m to v_n . That is, the probabilities are used to construct the operators for unitary evolution. Considering the action probabilistically, the average action is calculated, which can be achieved locally by $\sum_{v_m} P(v_n, v_m) S(v_n, v_m)$ and globally in the spin network Γ by considering that random walks on arbitrary vertices generate a global average action S_{Γ}

$$S_{\Gamma} = \sum_{v_n} \sum_{v_m} P(v_n) P(v_n, v_m) S(v_n, v_m).$$

$$\tag{45}$$

This is the total average action. To find the most efficient ratio between spacetime and particle building blocks, we must look for the probabilities that minimize the action. This is intuitive because probabilities are ratios. And the sum above implies the existence of a set of probabilistic matrices. It is interesting to note that both the Pauli and Dirac matrix formalisms can be reduced to binary matrices. The highest probability non-trivial eigenvalues for any $n \times n$ binary matrix are ϕ and $-1/\phi$. See Appendix (A). The precise definition of "non-trivial" depends on the dimension. At small dimensions (less than 10), the domination of ϕ is very clear. After dimension 50, additional "trivial" values can be the most probable with ϕ still holding a relatively high rank. Naturally, when the dimension goes to infinite, only 0, the singular trivial value, has a significant probability, all other eigenvalues going to a zero measure.

The efficiency of discrete quantum geometry can be exploited by looking at which of those matrices describe realistic evolution and minimize the average total action. In other words, find the matrix generating the minimal action using approaches which can be implemented by a partition function in the language of statistical mechanics or a quantum mechanics path integral formalism. At this level, it is the same object. So this is an efficiency and optimization action principle that can work for quantum gravity and for discrete versions of other force quantization models. Further investigation is planned to make clear how to derive, from the action principle, (45) the building blocks for the charge and volume quantization equations (43) and (44). However, for now, the above evidence implies that these foundational matrices are deeply related to the golden ratio. See Appendix (A). The code theoretic physics view, which implies an information theoretic universe, indicates that the choice of fundamental ratios and building blocks must be guided by the notion of efficiency or computational least action [63]. This leads naturally to the golden ratio with its powerful properties of being deeply associated with icosians [64], gauge symmetry physics, closure under multiplication and division and the crucial fractal self-similarity quality where $\frac{A}{B} = \frac{A+B}{A}$, allowing for maximally connected quantum-topological non-deterministic graph codes. Interestingly, because symbolism traffics in the quantity called information or meaning and, considering the principle of efficient language [2], we see that the most principle may be one that takes into account the rank of geometric meaning, which can be considered as complexity of order. For example, in the entropic phase of some binary alloy, we can have a high entropy phase with low complexity of order when order is low. And, at the low entropic phase, we have high oder at the crystal phase but low complexity rank because the network of objects is a homogeneous arrangement with all near neighbor distances being equal. Code theoretically, this is an ordering of just one spatiotemporal symbol, which defines it as a non-code. The maximum rank of complexity order is at the non-zero limit of spatiotemporal symbols, which is 2. The only way to achieve this in a network is by compositing a network of 1D two-letter codes known as Fibonacci chains make of spacings that are 1 and the golden ratio.

 \hookrightarrow If reality is code theoretic, its purpose is to efficiently express meaning, such as physical information, with its ultimate conserved quantity-quantized actions of the evolving substrate. Specifically, syntactically free binary choices in the self-emergent code theoretic network. Efficiency is achieved by (1) operating as a neural network code that generates maximal meaning from binary actions and (2) strategically placing those choices for maximal meaning (generally, physical meaning) according to syntactically free code choices.

4.3 Golden ratio

From the information theoretic and code theoretic views, particles, motions, interactions and spacetime geometry are language based. This new paradigm implies a non-arbitrary code operating at the Planck scale. We suggest the problem of quantum gravity is the search for the discovery of this non-arbitrary and efficient code of nature. With respect to physical codes, much progress has been made in understanding quasicrystalline codes. The most famous is the DNA code [65, 66], a golden ratio based quasicrystal biomolecule. Quasicrystal codes in materials science have also been studied [15, 42, 43, 67, 68, 69, 64, 70].

In 1944, Schrödinger predicted that the molecule encoding life would be a "quasiperiodic crystal" before the term quasicrystal was coined in the 1970s [71]. Amazingly, this was well before the precise molecular identification by Crick and Watson [66]. Each of the two DNA strands is a 5-periodic helix exactly expressed with the golden ratio. The two strands offset along the shared helical axis by two sequential Fibonacci numbers closely approximating the golden ratio. DNA has rotational symmetry but not translational symmetry—one of the defining characteristics of quasicrystals. DNA is a 3D network of deep and narrow double well potentials allowed by golden ratio based atomic organization. This aperiodic discrete energy landscape is part of what defines DNA code, where efficiency is achieved via the maximum spatiotemporal restriction of atoms without being fully restricted, as opposed to a crystal wherein all energy wells of the same size are occupied in each atom is locked into an EM trap. That is, a quasicrystal is a network of double well potentials with some wells occupied and others vacant, wherein spatiotemporal freedom of the atoms approaches the non-zero limit. For example, unlike a crystal, the assembly rules for a quasicrystal allow self-organization or construction choices within the rules that are not forced, i.e., non-deterministic. And dynamically, the orderly pattern of vacant energy wells allows particle dynamics not possible in crystalline or amorphous phases of matter. Due to the fractal self-similar quality where $\frac{A}{B} = \frac{A+B}{A}$, the non-zero limit of spatiotemporal freedom can only be achieved via golden ratio spacing ratios

between energy wells.

A quasicrystal is a structure that is ordered but not periodic. It has longrange quasiperiodic translational order and long-range orientational order. It has a finite number prototiles or "letters". And it has a discrete diffraction pattern indicating order but not periodicity. Mathematically, there are three common ways of generating a quasicrystal: the cut-and-project method (projection of an irrational slice of a higher dimensional crystal) [15], the dual grid method [15], and the Fibonacci grid method [42, 43]. Finite quasicrystals can be constructed by matching rules. Quasicrystals were discovered in nature via synthesis only in 1984 [72].

A hallmark and general characteristic of the 300 or so quasicrystals physically discovered is the golden ratio. Most of these quasicrystals are projections or subsets of the Elser-Sloane quasicrystal [65], which is a cut-andproject of the E_8 lattice using the angle $\arctan(1/\phi^3) \approx 13.28$ and or the angle $\pi/4 - \arctan(1/\phi^3) \approx 22.24$. The simplest quasicrystal possible is the two length Fibonacci chain, as 1 and $1/\phi$.

The Penrose tiling, a 2D quasicrystal, is a network of 1D quasicrys-3D quasicrystals in nature, such as a 3D Penrose tiling (Ammann tals. tiling) are networks of 2D quasicrystals, which are each networks of 1D quasicrystals-generally Fibonacci chains. So the irreducible building block of all quasicrystals are 1D quasicrystals. The "letters" of these 1D spatiotemporal codes are lengths between vacant or occupied energy wells. And a 1D quasicrystal can have any finite number of letters. However, the minimum is two. The Fibonacci chain is the quintessential 1D quasicrystal. It possesses two lengths related as the golden ratio. There are an infinite number of 2-letter 1D quasicrystals. However, maximally code efficient quasicrystals in higher dimensions that utilize only two letters must be constructed from Fibonacci chain 1D quasicrystals with letters being 1 and the inverse of the golden ratio as uniquely generated by a cut-and-project of the Z_2 lattice, in the direction of $\arctan(1/\phi^3) \approx 13.28$ degrees from the diagonal direction of the unit cubic cells.

Returning to the principle of efficient ratios and quantization conditions (30) and (38), we note the complementary role of the dual space in the case of discrete structures like DNA and quasicrystals in general. In (30) and (38), this relationship is obvious⁹. This dual space can be derived from the first object by generating its diffraction pattern. The details of the duality and quantization condition depend on the specific physics considered. The diffraction pattern is quantized in relation to the distance between physical objects in physical space. The interesting case is when distances are irrational numbers such that the ratios allow aperiodic order evidenced by the discrete Bragg peaks in the dual space as the diffraction pattern. Again, when the irrational number is the golden ratio, values are closed under multiplication and division and the ratio itself is fractally self-similar, which allows the maximally connected or densest network of 2-letter quasicrystals to be organized

⁹Let \mathcal{L} be a lattice. All lattices \mathcal{L} have an associated dual lattice \mathcal{L}^* , the set of vectors \vec{y} whose scalar products with the vectors \mathcal{L} , \vec{x} are integers, $\vec{x} \cdot \vec{y} = n$.

in dimensions 2D to 4D—the convergence of networks of 1D Fibonacci chain codes into higher dimensional quasicrystal codes.

 \hookrightarrow Let us bring together some of the above ideas to support the conjuncture that the coupling constant at the Planck scale is $1/\phi^3$ –the quantization of charge and the proposed discretization of spacetime. Simplex integers, as shape-numbers, are the fundamental building blocks of the QSN quasicrystal (a 3D network of Fibonacci chains) [14]. The principle of efficiency of ratios or the more general principle of efficient language states that fundamental ratios allow maximally efficient geometric codes, specifically quasicrystals. We presented the prominence of the golden ratio in two important natural codes—the DNA quasicrystal and metallic quasicrystals.

Next, we shall present some results in fundamental physics that may be clues that this direction is correct.

4.4 Golden ratio in physics

In this section, we review results that come from different fields of physics. All indicate the preeminence of the golden ratio. These results suggest that, with theoretical and experimental advances, the role of the golden ratio will become more clear.

4.4.1 Atomic physics-the hydrogen atom

The hydrogen atom is important because it is the simplest and most abundant element, having a single electron and nucleon. It serves as the foundation for all atomic theory. With the hydrogen atom, we have a case where the electron and atomic charge are identical, allowing us to isolate fundamental atomic structure ratios to test for the principle of efficient ratios. First, let us consider the hydrogen spectral series. When one electron goes from a higher to lower energy state, spectral emission occurs. The wavelengths of emitted/absorbed photons is given by the Rydberg formula in vacuum

$$\frac{1}{\lambda_{\rm vac}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right),\tag{46}$$

where R is the Rydberg constant

$$R = \frac{\alpha^2 m_e c}{4\pi\hbar} = \frac{\alpha^2}{2\lambda_e},\tag{47}$$

and n_1 and n_2 are integers greater than or equal to 1 such that $n_1 < n_2$ corresponding to the principal quantum numbers of the orbitals occupied before and after the quantum leap. For example, with $n_1 = 1$, we have the Lyman series [73], which we can rewrite as

$$\lambda_{vac} = \frac{1}{R} \left(\frac{n_2^2}{n_2^2 - 1} \right) = \frac{2\lambda_e}{\alpha^2} \left(\frac{n_2^2}{n_2^2 - 1} \right), \tag{48}$$

where λ_e is the Compton wavelength of the electron. It gives for $n_2 = 2$. For example, $1.2156 \times 10^{-7} m$. If we use an approximation as $\alpha = \frac{1}{20\phi^4}$

$$\lambda_{vac} = 800\phi^8 \lambda_e \left(\frac{n_2^2}{n_2^2 - 1}\right) = 800(13 + 21\phi)\lambda_e \left(\frac{n_2^2}{n_2^2 - 1}\right), \quad (49)$$

we get $1.2158 \times 10^{-7}m$. As expected, the first excited state of the hydrogen atom is ϕ -based [74]. When we add corrections of the energy levels of the hydrogen atom due to relativity theory and spin-orbit coupling, the hydrogen fine structure shows up. The fine structure is α -based and ϕ -based.

4.4.2 Standard Model: Neutrino Mixing and the Cabibbo angle

The first evidence for physics beyond the standard model (SM) of particle physics comes from experimental neutrino oscillations. See [75] and references therein for a review. One case of experimental evidence is the detection of solar neutrinos. Only electron neutrinos are emitted by the sun. However, only about 30 percent of the number predicted by theories that explain how the sun works are actually measured. This disagreement was resolved by an improved understanding of neutrino physics. Specifically, electron neutrinos are converted into muons and tau neutrinos. This is in agreement with various experiments. Neutrino oscillation is a verification that neutrinos are massive, which is in disagreement with the SM, where neutrinos are expected to be massless. So the understanding from experiment is that there is a lepton mixing matrix U—the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix U_{PMNS} , which relates the basic SM neutrino states, ν_e , ν_{μ} , ν_{τ} , related with the electron, muon and tau to the neutrino mass states ν_1 , ν_2 , and ν_3 with masses m_1 , m_2 , and m_3 .

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$
(50)

This is similar to the usual CKM (Cabibbo–Kobayashi–Maskawa) quark mixing matrix. The lepton mixing matrix U_{PMNS} has large uncertainties. Despite the lack of a deep knowledge of U_{PMNS} , the aforementioned data leads to an approximation of the first order, which allows for a theoretical description. There are different patterns of the values and parameterizations for this matrix that are in agreement with experimental data¹⁰. We suspect that the most powerful mixing matrix models are golden ratio based, as shown in [76] and was further developed in [77, 78, 79, 80], the so called golden ratio prediction. But more important than the exact values of the numbers that constitute the matrix is the idea that they can be derived via first principles from the symmetry breaking relationships of fundamental symmetries at the unification scale relating to hyper-dimensional lattices and their associated Lie algebras. For example, one can use the rotational icosahedral group [78],

¹⁰Different constructions of this matrix can be found in [75].

the alternating group of five elements A_5 [79] or $A_5 \times Z_5 \times Z_3$ [80]. And one can recover gauge symmetry physics from an icosian construction of the E_8 lattice [55]. Of course, by definition, all these groups are golden ratio based.

In this context, we have the important concept of quark-lepton complementarity which proposes that one parameter of the U_{PMNS} matrix, the solar angle (θ_{12}), is related with one parameter of the quark matrix mixing, the Cabibbo angle (θ_c), by

$$\theta_{12} + \theta_c = \frac{\pi}{4}.\tag{51}$$

We conjecture that the Cabibbo angle is the universal parameter which controls the entire structure of fermion masses and therefore appears in many places, such as mass ratios and mixing parameters. Recent data and theoretical proposals indicate the possibility of $\theta_c \approx 13.28$ [75, 77, 81, 82, 83]. This fits well with the golden ratio prediction [77]. The findings of these authors is not surprising because, as mentioned earlier, this angle appears in the cut-and-project of E_8 to 4D, breaking the crystal symmetry of E_8 to icosian associated H_4 symmetry. Similarly, this angle is required to bring the projection of E_8 to 3D into H_3 symmetry.

Both the 4D and 3D projections are golden ratio based quasicrystals. Similarly, the Weinberg angle (θ_W) of the electroweak interaction is an unknown value experimentally suggested to be slightly less than ~ 30°. In practice, what is measured is the quantity $sin^2(\theta_W)$. In a study of parity violation in Møller scattering a value of $sin^2(\theta_W) = 0.2397 \pm 0.0013$ was obtained at momentum transfer, Q = 0.16 GeV/c, establishing experimentally the "running" of the weak mixing angle. LHCb measured, in 7 and 8 TeV proton-proton collisions, an effective angle of $sin^2(\theta_W) = 0.23142$, though the value of Q for this measurement is determined by the partonic collision energy. The mean of these measured values deviates from $1/\phi^3$ at five 10,000 ths. The measurement uncertainty of 0.2397 is 0.0013, the 1,000 ths part, so our conjecture is well within the experimental margin of error.

To understand the origin of fermion masses and mixtures in the SM, one must turn to the family of symmetries that restrict the form of the mass generator matrices and models the hierarchies and mixtures through small symmetry breaking perturbations. Recent investigations, such as in the above references, indicate that icosahedral symmetry acts as an important scale unifier of the electromagnetic, weak and strong forces ($\sim 10^{25} eV$). If flavor has an underlying simplicity associated with unification symmetry, it would be productive to attempt recognition of it from neutrinos instead of from charged leptons and quarks, which have more complicated structures. This suite of experimental and theoretic evidences for massive neutrinos and our theoretical proposal of quasicrystal symmetry in unification scale physics supports the conjecture of a quasicrystalline spacetime and particle structure at the Planck scale.

We have found fundamental golden ratio values in the structure of E_8 itself. The root vector polytope of E_8 is the Gosset polytope¹¹. One of the

¹¹The E8 root system contains 240 root vectors spanning \mathbb{R}^8 , each with the same length

building blocks of E_8 is the 3-simplex, a regular tetrahedron. For example, the E_8 lattice and the Gosset polytope can be constructed, if one chooses, solely from tetrahedra. Two face-kissing tetrahedra in E_8 live in the same 4D subspace of R^8 . And certain pairs share one vertex that is the centroid of the Gosset polytope while their other vertices are the vertices of the Gosset polytope. The following table 4.4.2 shows the list of the cosine values of the angles, most of which are golden ratio expressions. Notice that the angle $ArcCos\frac{1}{4}$ can be written as $\pi/3 - ArcCos - \frac{3\phi-1}{4}$. It is worth noting that every eight of these tetrahedra live in the same 3-space (grouped into two dual orientations) and can be extended to an A_3 lattice. Since, the Gosset polytope can be generated by composition of all the transformations of one tetrahedron through these angles, the E_8 lattice can be generated via composition of the rotational transformation of the A_3 lattices through these angles.

| $\frac{1}{2}$ | $\frac{7}{(4\phi-2)^2}$ | $-\frac{3}{8\phi - 4}$ | $\frac{1}{4}$ | $\frac{1}{2(2\phi-1)}$ | $\frac{2\phi-1}{4}$ |
|---------------|-------------------------|------------------------|---------------|------------------------|---------------------|
|---------------|-------------------------|------------------------|---------------|------------------------|---------------------|

The Cartan matrix of the E_9 Lie affine algebra and the E_8 Lattice is:

$$C_{E_9} = \left(\begin{array}{cccccccccccccccccc} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right)$$

Its 9 eigenvalues are $4, \phi + 2, 3, \phi^2, 2, 2 - \phi^{-1}, 1, \phi^{-2}, 0$ and form the diagonal matrix Λ_{E_9} . The last eigenvalue is 0 because the Lie algebra is affine. Five of the eigenvectors (columns of V_{E_9}) have integer coordinates and the four other have coordinates which are 0 or a power of ϕ . We note φ for the inverse of ϕ .

and forming the vertices of the Gosset polytope.

$$V_{E_9}^{-1} = D_{E_9} V_{E_9} \tag{54}$$

$$C_{E_9} = V_{E_9} \Lambda_{E_9} D_{E_9} V_{E_9} = V_{E_9} \Lambda_{E_9} V_{E_9}^{-1}$$
(55)

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 2 & -\phi^2 & 1 & -\varphi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & \varphi^2 & 0 & -\phi^2 & 3 \\ 4 & -\phi & -1 & \varphi & 0 & \varphi & -1 & -\phi & 4 \\ -5 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 5 \\ 6 & \phi & 0 & -\varphi & 0 & -\varphi & 0 & \phi & 6 \\ -3 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 3 \\ -4 & -\phi & -1 & -\varphi & 0 & \varphi & 1 & \phi & 4 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{vmatrix} 4 \\ \varphi^2 \\ -2 \\ \varphi^2 \\ 0 \\ 0 \end{vmatrix} \begin{pmatrix} -1 & \phi & -1 & \varphi & -1 & -\varphi & 1 & -\phi & 1 \\ 2 & -\phi^2 & 1 & -\varphi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 1 & -\phi^2 & 2 \\ -3 & \phi^2 & 0 & -\varphi^2 & 0 & -\phi^2 & 3 \\ -5 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 5 \\ -3 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 3 \\ -4 & -\phi & -1 & -\varphi & 0 & \varphi & 1 & \phi & 4 \\ 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Equation 54 gives a simple expression of the inverse of the eigenvector matrix. Equation 55 states the diagonalization of the Cartan matrix. Equation 56, where the formal notation $\lfloor v \rfloor$ expresses the diagonal matrix form of the vector v, (here $\lfloor v \rceil = \Lambda_{E_9}$), is unexpectedly a new formula not previously reported in the literature.

Our intent was to look for a result in E_8 similar to that found by Sirag in the relationship between E_7 and $C_{E_7^+}$ [84]. We analyzed the spectrum of the C_{E_9} Cartan matrix to obtain our result- a systematical study of the spectrum of all Cartan matrices though the Dynkin diagram of all simple Lie algebras and their affine extensions. We found that the golden ratio appears in the eigenvalues, outside of the exceptional $E_9 = E_8^+$, only for SU(5k), where k is an integer. This crucial emergence of ϕ from the E_8 lattice may be the key explanation of three other facts:

- McKay correspondence [84] between E₈ Lattice and double icosahedral group
- Correspondence between icosians and E_8 Lattice
- Discrete Hopf fibration $S^3 \to S^7 \to S^4$ [85]

We can now formulate our ontological theorem:

Theorem 4.1. E_8 generates the golden ratio.

Proof. The proof is given in equations 55 and 56.

Proposition 4.2. We propose the following ontological chain: $\mathbb{B} \Rightarrow \mathbb{T} \Rightarrow E_8 \Rightarrow \phi \Rightarrow \mathbb{D} \Rightarrow QC \Rightarrow \mathfrak{h}_{92} \oplus \mathfrak{a}_7 \Rightarrow QSN$

- The bits 0 and 1, or equivalently, the sphere S^0 emerges ab initio
- The trits, the integer, the matrix structure and the structured division algebra emerge from the bits, as described in Annex A (which also suggests a direct path to φ)
- The last division algebra is the octonions. Its integers, the Cayley integers, emerge as the E_8 lattice
- From theorem 4.1, the golden ratio ϕ emerges from the E_8 lattice
- Extending the integer by ϕ , our ring \mathbb{D} of Dirichlet integers and its structural tensor products quantize all the needed algebras
- From the E₈ lattice, and in a simple way by using explicitly φ or in a more subtle way implicitly, emerges the Elser-Sloane quasicrystal
- Spacetime geometry emerges from this structure on which a gauge algebra is naturally inherited from the E₈ Lie group
- A 3D representation made of only regular tetrahedra forming a quasicrystal code made of Fibonacci chains, the QSN, emerges to support the geometrical reality

Proof. We do not yet have a proof of this statement. However, each of the eight steps in the proposed ontological chain is mathematically sound and supported by evidence. The logical flow is strong. Accordingly, this is more than a conjecture. It is a deduced hypothesis. Readers are invited to think critically. This mathematical first-principles based emerging theory is incomplete but yet evidenced as the right direction via the physical evidence and equations presented herein. $\hfill \Box$

4.4.3 Quantum theory

An important result in foundational quantum mechanics is that of Hardy [86], which can be considered the best version of Bell's theorem [87] because of its simplicity and logical/mathematical rigor. While Bell's proof of the impossibility of Einstein, Podolsky and Rosen's view is based on statistical predictions and inequalities, Hardy's test is a test of non-locality without inequalities.

This test considers two qubits prepared in an entangled state, wherein each is sent in the opposite direction by a source in the middle of two detectors. One electron goes to the detector at left and the other to the right. The entanglement can be on polarization, spin components or momentum. The detectors have a switch between two positions that can be settled randomly and independently after the particles are emitted by the source and before it arrives at the detectors. We call the detector on the left L and on the right R. So they can have two positions L1, L2 and R1, R2. The choice for the switches are experiments with two possible outcomes, say ± 1 , that depend on entangled particles. Just four of the possible experiments are needed to get the result of maximal non-locality, which we will present herein. These four experiments are fixed positions of the detectors to L1, R1 (let us call this experiment A); L1, R2 (experiment B), L2, R1 (experiment C) and L2, R2 (experiment D). Running these experiments many times, shows that, in experiment A, L1 and R1 being both -1 never occurs. In experiment B, with L1, R2, and experiment C, with L2 and R1, both being +1 never occurs. And in experiment D, L2 and R2 being both +1 occurs sometimes. According to local determinism, suppose that the atom going left has instructions to come out +1 if it encounters position L2. According to experiment C, if L2 gives +1, R1 can't give +1. So it must give -1. According to experiment A, if R1 gives 11, then L1 must give +1. And according to experiment B, if L1 gives +1, then R2 must give -1. This instruction set is the only one consistent with predictions A, B, and C and with L2 giving the result +1. It shows that, in situation D (for which the detectors are set to L2 and R2), whenever the left atom comes out +1, the right qubit must be -1. However, according to quantum mechanics, there is a probability for the right qubit to come out as +1. The exact probability P for L2 and R2 being both +1 is

$$P = \phi^{-5}.$$
 (57)

This result is in strict accordance with both quantum mechanics experiment [88]. That is, the probability for the simplest quantity relationship, two, of the simplest fermions, electrons, in the simplest possible dynamic and geometric relationship (moving apart at equal velocities on the same line) is a probability of the golden ratio to the power of -5. This result is deep and important because, in some sense, this is the probability for maximal non-locality in nature.

4.4.4 Chaos theory: KAM theorem

The Kolmogorov, Arnold and Moser (KAM) theorem¹², which is a result in classical mechanics of the study of dynamical systems. It is about the persistence of quasiperiodic motions under small perturbations. A dynamical system can be expressed in configuration space in terms of action-angle coordinates consisting of action and angle variables in terms of a torus defined by its angle variables. Quasiperiodic orbits represent integral motion on a torus and, if it is integrable, there is a constant or invariant of the motion associated with the torus that leads to the term *invariant tori*. What happens to the invariant tori as the nonlinearity of the system increases? Say we have a system with Hamiltonian $H = H_o + \varepsilon H_1 + ...$, where H_o is the non-perturbed Hamiltonian and H_1 the non-linear perturbation allowing ε to mediate the force of the perturbation. For sufficiently small perturbations, virtually all tori are preserved.

Consider the frequency of motion around each angular variable of a torus. As a point moves, it rotates around the tube while revolving around the torus axis. If we take the ratio of these frequencies, we get a quantity called the

¹²There are many references in the context of chaos theory. For example, see [89, 90].

winding number, $\sigma = \frac{\omega_1}{\omega_2}$. The KAM theorem shows that the tori most easily destroyed are those with rational winding numbers, while almost all other orbits (those with irrational winding numbers) are preserved. What happens if we increase the perturbation? The KAM theorem itself doesn't explicitly say, but derivative work does [91]. After the rational winding number tori go chaotic, the irrational tori eventually break up also, even though they are significantly more robust under perturbation. As the perturbation grows, more irrational tori go unstable. Most interestingly, the tori go unstable in order of the degree of irrationality of their the winding numbers. This is where golden ratio shows up as a physical limit. Mathematicians consider the golden ratio to be the most irrational of numbers because the rank of a number's irrationality is based on the speed of convergence of its continued fraction expression. The continued fraction expression for the golden ratio uses only the integer 1, making it the most "difficult" or slowest number to approximate with rational numbers. It approaches the limit slower than any other continued fraction expression.

It is at this golden ratio based winding number where physical vortices are most stable under increasing perturbations. This is the special limit related to how small denominators correspond to the growth of ε , allowing it to be minimal. It is the state where the conditions for the KAM theorem are most easily satisfied. In summary, golden ratio based gravitational, fluidic and other dynamical vortices are the most stable and therefore the most statistically probable physically abundant in nature.

4.4.5 Black hole physics

Black hole physics is a good laboratory for testing the limits of general relativity and quantum mechanics and for putting the two limits together in the study of quantum gravity theories. Accordingly, let us look for clues about Planck scale golden ratio physics in black hole equations. The first result is within the context of non-arbitrarily limiting and manipulating the equations of general relativity [92, 93]. This is a classic general relativity result with no relation to quantum mechanics. A rotating black hole can have a transition of phase between positive specific heat capacity and negative. The transition is based on the golden ratio when the ratio of angular momentum J and mass M is kept constant in the equation. In this case, ϕ is the point where a black hole's modified specific heat changes from positive to negative

$$\frac{M^4}{J^2} = \phi. \tag{58}$$

This non-arbitrary manipulation is analogous to the artificial but non-arbitrary setting of the velocity of the electron to zero in order to derive the electron rest mass. That is, although there is no such thing as an electron at rest, the non-arbitrary but non-physically realistic manipulation of the electron equation provides a deep fundamental understanding of a limit in nature. Another result in classical general relativity which indicate special role of golden ratio in underlying geometry of spacetime was reported recently in [94]. In this case the golden ratio appears in the rather simple field of Schwarzschild black holes with a cosmological constant. Let us now combine quantum mechanics with general relativity by considering the loop quantum gravity approach to quantum gravity [1]. Using this work done on the microstructure of spacetime, we compute black hole entropy. In a simplified framework, the isolated quantum horizons formulation of loop quantum gravity, we can derive the lower bound of black hole entropy [28]

$$e^{\frac{8\pi\hbar GS}{kA}} \ge \phi,\tag{59}$$

where S is the entropy, A the black hole area and k the Boltzmann constant. In section (3.1), we go further, using arguments from information theory to fix the loop quantum gravity parameter, which we can wright as

$$2^{\pi\gamma} = \phi. \tag{60}$$

5 Discussion

The (1) principle of least action and Noether's second theorem, (2) gauge symmetry, (3) general relativity and (4) the implication of spacetime discretization at the Planck scale strongly suggest an underlying new principle based on the efficiency of ratio, where, mathematically, the golden ratio is a special limit in the universe of all ratios and where, experimentally, it is observed in the fundamental physical equations and limits of nature.

Assuming that reality is discrete at the Planck scale, a logically satisfying explanation for the observed quantization of charge leads us to ask, "What is the most efficient organizing principle for how dynamical charge space is constructed?". A discrete reality is ultimately numerical, although not necessarily digital. It may be based on self-referential volumetric shape-numbers, as 3-simplex integers forming a quasicrystalline graph drawing based description of quantum gravity.

This multifaceted overall argument strongly implies that reality is code theoretic. The natural geometric languages that we know, such as DNA and other quasicrystals, are defined by the golden ratio, serving as experimental cases where the golden ratio is used by nature in geometric codes and inspiring the axiomatic theory that the physical universe behave with optimal digital efficiency by exploiting our foundational dimensionless constant ultimately residing at the Planck volume of space. This golden ratio based running constant starts at the quantum gravity scale and goes upwards fractally (all quasicrystals are fractal). The well-known self-similarity properties of the golden ratio and quasicrystals explain the observed fractality of nature at all scales [62, 96, 97, 98, 99, 100, 95], and provide a posteriori credit to the Dodecahedron-Icosahedron doctrine [101].

Note that, assuming the discreteness of spacetime at the Planck scale, we face a problem similar to one at the atomic scale: How do these building blocks self-organize to build more complex structures? How does order emerge? A crystalline structure would be a naive answer because crystals are not codes. Conversely, quasicrystalline structures are codes with non-local long range order. Syntactical freedom in their static and dynamic ordering rules provide the freedom necessary for quantum non-determinism and the ability for physicists to recover the gauge symmetry unification equations of particles and forces that express in the code due to the mapping of Lie algebras to the generating hyper-lattices from which quasicrystals are created.

6 Conclusion

We have presented a compelling idea that we can apply the results and tools from quantum information and quantum computation to a quantum spacetime code theoretic view using algebraic graph drawing formalism. We considered a DQW of a quantum particle on a quantum gravitational field and studied applications of related entanglement entropy. This memory time based entanglement entropy drives an entropic force, suggesting a unified picture of gravity and matter. Following this, we proposed a model for walker positions topologically encoded on a spin network, which can easily be reexpressed using twistors. This results in anomaly cancellation because the particles are no longer points but Planck scale voxels, as tetrahedral units of spacetime.

Later in this document, we presented a review of results that serve as compelling clues about an underlying organizing principle related to charge quantization and the golden ratio. The appearance of the golden ratio in black hole physics, the indication from the research of physics beyond the standard model and other evidences suggest there is quasicrystalline geometry at the unification scale and allow for a logical argument that nature has a limit of non-locality in a two particle quantum system. Again, an element of evidence for quasicrystalline structure at the quantum gravity scale. Due to the inherent fractal scale invariance of golden ratio based quasicrystals, this new efficiency and organizing principle shows up at other scales, such as the atomic scale, where, for example, quasicrystalline structure is present in the hydrogen atom and DNA. This unifying principle connecting the discussed points correlates nicely with derivative works from the KAM theorem, where it is trivially true that literally all systems in nature are both propagating and rotating simultaneously-forming propagating gravitational and electromagnetic vortices or extruded tori. The quasicrystalline phase is a special limit in thermal dynamics, where spatiotemporal degrees of freedom reach the maximum non-zero limit.

7 Acknowledgements

We also would like to acknowledge Carlos Castro Perelman for useful suggestions of references.

A Binary Matrices

We show that all n-dimensional structures used in relativity and quantum mechanics reduce to products of binary matrices. The quaternionic and complex structures, and the Pauli and Dirac algebra are illustrated below. The process [13] involves replacing i by a 2x2 anti-diagonal anti-symmetric "trit" (-1,0,1) matrix and then further reducing trit matrices by substituting -1 by 2x2 anti-diagonal symmetric non-null binary matrices.

A.1 Negative numbers and complex numbers

Algebraic structures emerge when a set of elements which, (along with operations) constitute our algebra, obey fundamental symmetries, eventually expressed as algebraic rules or constraints. For example, from the set of positive numbers, 0,1,2,3,4..., we define a new element m, outside of this set, satisfying $(m)^2 = 1$. So m is not 1. Indeed m = -1, and we have defined the negative unit, which we call -1. We extend our set and define $\mathbb{Z} = \mathbb{N}1 + \mathbb{N}m$ by using linear combinations of our old unit (1) and our new unit (m=-1) with coefficients in the old set \mathbb{N} . We can also use 2x2 matrices $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $m = -1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and check that $(m)^2 = 1$. Therefore the eigenvalues satisfy the same constraint. The complex numbers have a similar representation $z = a1 + bi = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} +$ $b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where *i* satisfies $(i)^2 = -1$. The complex structure ture is the multiplicative group of the Gaussian integer units $\{1, -1, i, -i\}$. The trit to bit map μ from $\mathbb{T} = \{-1, 0, 1\}$ to $\mathcal{M}_2(\mathbb{F}_2)$ where \mathbb{F}_2 is the binary field {0,1} is defined by $\mu(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mu(-1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \mu(0) =$ $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and can be applied to each matrix element obtained from the complex units by the map ν from the fourth roots of 1 to $\mathcal{M}_2(\mathbb{T})$ such that $\nu(i) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \nu(-i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \nu(0) = \mu(0), \nu(1) = \mu(1), \nu(-1) = \mu(1), \mu(1), \mu(1) = \mu(1), \mu(1), \mu(1) = \mu(1$ $\mu(-1)$ to derive 4-dimensional binary matrices in $\mathcal{M}_4(\mathbb{F}_2)$ from the combined map $\rho = \mu \circ \nu$.

For example,
$$\rho(i) = \begin{pmatrix} \mu(0) & \mu(-1) \\ \mu(1) & \mu(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Note that in the two following subsections, the algebra of these matrices is not the standard algebra on $\mathcal{M}_n(\mathbb{R})$. It is also not the standard algebra on $\mathcal{M}_n(\mathbb{F}_2)$. The operations have to be computed in \mathbb{R} and then projected to $\mathcal{M}_{\frac{n}{2}}(\{0_2, I_2, m\})$ by a process described in [13], where $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ projects to 0_2 . In the alternative representation proposed on $\mathcal{M}_n(\mathbb{T})$, the projection is simply realized by the sign operator: $\forall x \in \mathbb{R}$, $\operatorname{sign}(x) =$

 $\begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}.$

A.2 Quaternions and Pauli matrices

It was shown in [13] how to use 4D trit maps in $\mathcal{M}_4(\mathbb{T})$ or 8D bitmaps in $\mathcal{M}_8(\mathbb{F}_2)$ to represent quaternions and implement the quaternion group (finite group of degree 8), the D_4 root vectors group of degree 24 and the F_4 root vector group of degree 48. We repeat here the results for quaternions to deduce the Pauli matrices. We can define as κ the following map from the positive unit quaternions $\{1, i, j, k\}$ to $\mathcal{M}_4(\mathbb{T})$:

$$\kappa(i) = \begin{pmatrix} \nu(-i) & \nu(0) \\ \nu(0) & \nu(i) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\kappa(j) = \begin{pmatrix} \nu(0) & \nu(1) \\ -\nu(1) & \nu(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

 $\kappa(i)$ is named yin and $\kappa(j)$ is named yang in [13] because they generate the ring of 24 Hurwitz integers [102] with the modified trit matrix algebra projecting back the result by applying the sign function and its geometric dual. Naturally, the map κ is completed because the matrices satisfy Hamilton's quaternion relations $ijk = i^2 = j^2 = k^2 = -1$. $\kappa(k) = \kappa(i)\kappa(j) =$

$$\begin{pmatrix} \nu(0) & \nu(-i) \\ \nu(-i) & \nu(0) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Quaternions are defined as binary matrices by applying μ to each element. This generates the map $\lambda = \mu \circ \kappa$ from $\{1, i, j, k\}$ to $\mathcal{M}_8(\mathbb{F}_2)$ and

Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ can be built by applying the map ρ to each of their elements $\{0, 1, -1, i, -i\}$:

We define a new imaginary:

0 0 0 1 0 0 0 0

The quaternions $\lambda(i), \lambda(j), \lambda(k)$ and the Pauli matrices $\Lambda(\sigma_1), \Lambda(\sigma_2), \Lambda(\sigma_3)$ are related by: $i\lambda(i) = \Lambda(\sigma_3), \lambda(j) = i\Lambda(\sigma_2), i\lambda(k) = \Lambda(\sigma_1)$. Together with $1_8 = \rho(1_2)$ and *i*, they form the bi-quaternion units.

The Clifford algebra $\mathcal{C}l_3$ also admits for basis in $\mathcal{M}_8(\mathbb{F}_2)$: $1_8, \Lambda(\sigma_1), \Lambda(\sigma_2), \Lambda(\sigma_3), \lambda(i), \lambda(j), \lambda(k)$ and *i*.

They are illustrated below, where 0 is gray and 1 is white:

| | |
|------|--|
| | |
| | |

A.3 Dirac matrices

The 16 Dirac matrices $\gamma_{a,b}$, where $(a,b) \in \{0,1,2,3\}$, $\gamma_{0,0} = I_{16}$, $\gamma_{0,b} = I_2 \otimes \sigma_b$, $\gamma_{a,0} = \sigma_a \otimes I_2$ and $\gamma_{a,b} = \gamma_{a,0}\gamma_{0,b}$ are given in the below table, where 0 is gray and 1 is white:



The same process can be extended based on $\lambda(i), \lambda(j), \lambda(k)$ or by combining the Dirac matrices with $\gamma_{0,i} = I_2 \otimes i, \gamma_{i,0} = i \otimes I_2$.

A.4 Binary matrix eigenvalues

These binary matrices have eigenvalues which are fourth roots of unity, the Gaussian integer units. We now systematically study all eigenvalues of ndimensional binary matrices $\mathcal{M}_n(\mathbb{F}_2)$. We begin with n=2. There are 16 matrices. Their eigenvalues are, with $\phi = \frac{1+\sqrt{5}}{2}$ and $\varphi = \phi^{-1} = \frac{-1+\sqrt{5}}{2}$:

(0,0), (1,0), (0,0), (1,0), (1,0), (1,1), ($\phi,-\varphi$), (1,0), (1,1), (1,0), (1,1), ($\phi,-\varphi$), (2,0) Unexpectedly, the only non-integer eigenvalues of $\mathcal{M}_n(\mathbb{F}_2)$ are ϕ and $-\phi^{-1}$. This is one of the deepest mathematical reasons for our overall thesis that the unique qualities of golden ratio serve as an ultimately efficient fundamental constant of physics, expressing a ratio that is code-theoretically more powerful than 1.

From the eigenvalues of the 512 matrices of $\mathcal{M}_3(\mathbb{F}_2)$, the most probable are by decreasing order 0 (32%), 1 (28%), ϕ (6%), $-\phi^{-1}$ (6%), 2 (5%), -1 (5%)... From the eigenvalues of the 65,536 matrices of $\mathcal{M}_4(\mathbb{F}_2)$, the most probable are by decreasing order 0 (25%), 1 (18%), -1 (5%), 2 (4%), ϕ (4%), $-\phi^{-1}$ (4%), $1 - \sqrt{2}$ (1%), ... From the eigenvalues of the 33 554 432 matrices of $\mathcal{M}_5(\mathbb{F}_2)$, the most probable are by decreasing order 0 (19%), 1 (11%), -1 (4%), 2 (2%), ϕ (1%), $-\phi^{-1}$ (1%), $1 - \sqrt{2}$ (1%), ... They are the roots of 8,927 characteristic polynomials of degree 5. When the matrix dimension n is grows, the most probable eigenvalues have decreasing probabilities. ϕ always remains in the group of the most probable because its characteristic polynomial has small coefficients (1 or -1). At the other end, the less probable eigenvalue is n and appears only once, for the matrix made of only ones and polynomials $nx^{n-1} - x^n$. These most probable eigenvalues are the centers of exclusion circles with no other values and with radii proportional to their probabilities. We also see a smaller exclusion interval on the real line as illustrated in figure 6. This repelling behavior between eigenvalues is a known indicator of universality [103].



Figure 6: $\mathcal{M}_5(\mathbb{F}_2)$ spectrum

All of the above binary structure matrices representing the quaternions, biquaternions, Clifford algebra, Pauli matrices and Dirac matrices have $\mathcal{M}_n(\mathbb{F}_2)$ sub-blocks which are in $\{\mu(0), \mu(1) \text{ and } \mu(-1)\}$. By using the nilpotent subblock $\epsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which has 0 as double-eigenvalue, we can implement dual integers. If we use the "golden" sub-block $\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, which has ϕ and $-\phi^{-1}$ as eigenvalues, we obtain an interesting structure. Note that in [77], the neutrino Majorana mass matrix $m_v = \begin{pmatrix} 0 & m & 0 \\ m & m & 0 \\ 0 & 0 & m_{atm} \end{pmatrix} = \begin{pmatrix} \underline{m\Phi} & 0 \\ 0 & 0 & m_{atm} \end{pmatrix}$ which is block if the two provides the structure.

 $\left(\begin{array}{c|c} m\Phi & 0\\ \hline 0 & m_{atm} \end{array}\right)$, which is block-diagonal with the sub-block Φ and therefore the mass eigenvalues make the neutrino-mixing angle $\theta_{12} = \arctan(\phi^{-1})$.

If a, b, c and d are integers, $(aI_2 + b\Phi)(cI_2 + d\Phi) = (ac + bd)I_2 + (ad + bc + bd)\Phi)$ and the Dirichlet integer $(a + b\phi)$ can be implemented as $(a\mu(1) + b\Phi)$.

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